A Restatement of Mathematical Considerations of TEM Modes on an n-Wire Line

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Abstract—A restatement of mathematical considerations of TEM modes on an n-wire line is presented. An n-wire line inside a shielding conductor or over a ground plane supports n independent TEM modes which can be determined by obtaining eigenvectors on the n-wire line deduced from the characteristic admittance matrix. It is shown conclusively that the TEM modes are determined by the geometrical arrangement of the n wires as well as by the manner of excitation on the n-wire line. Power division ratios on each wire and terminating admittances for output ports of each wire are also discussed, and it is shown that one can excite a TEM mode similar to an even mode and $n-1$ TEM modes, each of which resembles an odd mode, on the n-wire line.

NOMENCLATURE

$$[\eta]=\begin{bmatrix}\eta_{11} & -\eta_{12} & \cdots & -\eta_{1n}\\ -\eta_{12} & \eta_{22} & \cdots & -\eta_{2n}\\ \cdot & \cdot\ \cdot & & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\ \cdot & \cdot\\ -\eta_{1n} & -\eta_{2n} & \cdots & \eta_{nn}\end{bmatrix}$$

$[\eta]$ $n \times n$ real symmetric hyperdominant matrix (characteristic admittance matrix).

$$\eta_{ij} \geq 0 \qquad (i,j = 1,\cdots,n)$$

$$\eta_{ie} = \eta_{ii} + \sum_{i}^{j\neq i} (-\eta_{ij}) \geq 0$$

$[\zeta] = [\eta]^{-1}$ Characteristic impedance matrix.

$\lambda = j \tan \theta$ Transformed complex frequency parameter.

$c = \cos\theta,\ s = j \sin\theta$

θ Electrical length of the line section.

ITI Ideal transformer interconnection.

HPD Hybrid power divider.

$\boldsymbol{V}_o = [V_{1o}\cdots V_{no}]^t$ Voltage vector for input ports of n-wire line and for output ports of ITI.

$\boldsymbol{I}_o = [I_{1o}\cdots I_{no}]^t$ Current vector for input ports of n-wire line and for output ports of ITI.

$\boldsymbol{V}_l = [V_{1l}\cdots V_{nl}]^t$ Voltage vector for output ports of n-wire line.

$\boldsymbol{I}_l = [I_{1l}\cdots I_{nl}]^t$ Current vector for output ports of n-wire line.

$\boldsymbol{V}_a = [V_{1a}\cdots V_{ma}]^t$ Voltage vector for input ports of ITI.

$\boldsymbol{I}_a = [I_{1a}\cdots I_{ma}]^t$ Current vector for input ports of ITI.

$[T]$ Transformation matrix of ITI.

G_o Conductance.

$[A] = \mathrm{diag}\,[a_1,\cdots,a_n] \qquad a_i > 0,\ (i = 1,\cdots,n)$

$$a_1^2 + \cdots + a_n^2 = 1.$$

$\boldsymbol{U}_i = [U_{1i}\cdots U_{ni}]^t$ Voltage vector for ith basic TEM mode.

$\boldsymbol{J}_i = [J_{1i}\cdots J_{ni}]^t$ Current vector for ith basic TEM mode.

$\boldsymbol{U}_i' = [U_{1i}'\cdots U_{ni}']^t$ Voltage vector for ith modified TEM mode.

$\boldsymbol{J}_i' = [J_{1i}'\cdots J_{ni}']^t$ Current vector for ith modified TEM mode.

E_n $n \times n$ unit matrix.

$\alpha_1,\cdots,\alpha_n$ Eigenvalues for $[\eta]$.

$\boldsymbol{P}_k = [p_{1k}\cdots p_{nk}]^t$ Orthonormal eigenvector for $[\eta]$.

$\beta_1,\cdots,\beta_n$ Eigenvalues for $[A]^{-1}[\eta][A]^{-1}$ and $[A]^{-2}[\eta]$.

$\boldsymbol{P}_k' = [p_{1k}'\cdots p_{nk}']^t$ Orthonormal eigenvector for $[A]^{-1}[\eta][A]^{-1}$.

$\boldsymbol{Q}_k = [q_{1k}\cdots q_{nk}]^t$ Orthonormal eigenvector for $[A]^{-2}[\eta]$.

$\boldsymbol{P}_u = 1/n^{1/2}[1\cdots 1]^t$.

$\boldsymbol{Q}_u = 1/n^{1/2}[1\cdots 1]^t$.

η_e Positive constant.

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I. INTRODUCTION

THE n-way HPD described by Wilkinson [1] splits an input signal into n equiphase and equiamplitude output signals. Parad and Moynihan [2] presented two-way HPD's with the output signals in phase and with arbitrary amplitude ratios. Cohn [3] presented broad-band two-way HPD's with equiamplitude output signals. Yee *et al.* [4] presented broad-band n-way HPD's with equiamplitude output signals. Ekinge [5] presented broad-band two-way HPD's with arbitrary output signals. Tetarenko and Goud [6] presented an n-way HPD with arbitrary output signals in the Appendix of their paper. All these papers synthesize TEM-mode HPD's. The objective of this paper is to present a restatement of mathematical consideration of TEM modes on an n-wire line.

The n-way HPD's should be designed by making use of multiwire lines which include not only multicoupled lines but also uncoupled lines. An n-wire line inside a shielding conductor or over a ground plane supports n independent TEM modes. It is discussed in Section II that the n independent TEM modes can be determined by obtaining eigenvector on the n-wire line deduced from the characteristic admittance matrix. It is shown in Section II that a set of TEM modes is determined by the geometrical arrangement of the n wires as well as by the manner of excitation on the n-wire line.

In order to establish synthesizing methods of the n-way HPD's, it is necessary to know power division ratios on each wire of the n-wire line and how to choose ratios among terminating admittances for outputs of each wire. The problem of power division ratios is discussed in Section II, and the ratios among terminating admittances in Section III.

Many papers analyze and synthesize n-way HPD's including two-way HPD's by making use of an even- and an odd-mode equivalent circuit [3]–[5]. It is discussed in Section IV that one can excite a TEM mode similar to an even mode and $n-1$ TEM modes, each of which resembles an odd mode, on the n-wire line.

II. TEM MODES ON LOSSLESS TRANSMISSION MULTIWIRE LINE

This section describes TEM modes on a lossless transmission multicoupled or multiwire line and an ITI that excites such TEM modes on the multiwire line.

A. Equations of Transmission of a Multiwire Line [7]

The following assumptions will be introduced on any multiwire line comprising the circuit under consideration.

1) All constituent conductors are perfectly conductive.

2) The dielectric space surrounding the conductors has uniform dielectric constant ϵ and permeability μ, and is perfectly lossless.

3) The arrangements of the conductors and dielectric are uniform in the direction along the line.

The transmission equations for a multiwire line are well known, and if the line satisfies the preceding three contions, then the propagation of waves on the line can be represented by a real symmetric characteristic admittance matrix $[\eta]$ or a characteristic impedance matrix $[\zeta]$ and the transformed complex frequency parameter λ, and the transmission equations reduce merely to the matrix extension of those for coaxial lines. An n-wire line is an (n,n) port as shown in Fig. 1, and its transmission equation

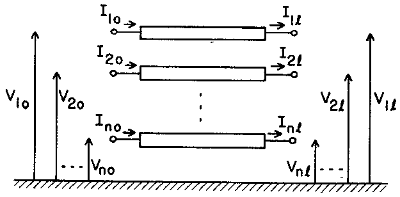

Fig. 1. An n-wire line.

can be expressed as

$$\boldsymbol{V}_o = c\boldsymbol{V}_l + s[\zeta]\boldsymbol{I}_l \tag{1a}$$

$$\boldsymbol{I}_0 = s[\eta]\boldsymbol{V}_l + c\boldsymbol{I}_l \tag{1b}$$

where

$$\boldsymbol{V}_o = \begin{bmatrix} V_{1o} \\ V_{2o} \\ \cdot \\ \cdot \\ \cdot \\ V_{no} \end{bmatrix}, \quad \boldsymbol{V}_l = \begin{bmatrix} V_{1l} \\ V_{2l} \\ \cdot \\ \cdot \\ \cdot \\ V_{nl} \end{bmatrix}, \quad \boldsymbol{I}_o = \begin{bmatrix} I_{1o} \\ I_{2o} \\ \cdot \\ \cdot \\ \cdot \\ I_{no} \end{bmatrix}, \quad \boldsymbol{I}_l = \begin{bmatrix} I_{1l} \\ I_{2l} \\ \cdot \\ \cdot \\ \cdot \\ I_{nl} \end{bmatrix}$$

$$c = \cos\theta \qquad s = j\sin\theta \qquad \lambda = j\tan\theta.$$

$[\eta]$, $[\zeta]$ is the $n \times n$ symmetric matrix, $[\eta] = [\zeta]^{-1}$ and θ is the electrical length of the line section.

The characteristic admittance matrix $[\eta]$ is represented as

$$[\eta] = \begin{bmatrix} \eta_{11} & -\eta_{12} & \cdots & -\eta_{1n} \\ -\eta_{12} & \eta_{22} & \cdots & -\eta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\eta_{1n} & -\eta_{2n} & \cdots & \eta_{nn} \end{bmatrix}. \tag{2}$$

Let's define $\eta_{ie} (i = 1,\cdots,n)$ as

$$\eta_{ie} = \eta_{ii} + \sum_{j}^{j \neq i} (-\eta_{ij}). \tag{3}$$

In case of $i > j$, η_{ij} becomes η_{ji}. This matrix $[\eta]$ is hyperdominant, that is

$$\eta_{ij} \geq 0, \qquad (i,j = 1,\cdots,n) \tag{4a}$$

$$\eta_{ie} \geq 0, \qquad (i = 1,\cdots,n). \tag{4b}$$

B. Ideal Transformer Interconnection

It is well known that an (m,n)-port ITI, shown in Fig. 2, transforms voltages and currents according to the following expressions by using a transformation matrix $[T]$:

$$\boldsymbol{V}_a = [T]^t \boldsymbol{V}_o \qquad [T]\boldsymbol{I}_a = \boldsymbol{I}_o \tag{5}$$

where $\boldsymbol{V}_o$, $\boldsymbol{I}_o$, given by (1)

$$\boldsymbol{V}_a = [V_{1a},\cdots,V_{ma}]^t \qquad \boldsymbol{I}_a = [I_{1a},\cdots,I_{ma}]^t, \qquad m \leq n.$$

$[T]$ is an $n \times m$ real matrix, and $[T]^t$ shows the transpose of $[T]$.

It is well known that one can transform a multiport network (for example, multiwire transmission line) into a diagonal network by using ITI. That is, an ITI can be used as a decoupling circuit among input ports of multiport network. An ITI can also be used as a power divider which distributes an input signal from an input port among output ports.

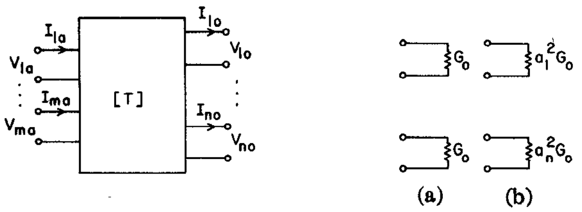

Fig. 2. ITI and conductances for output ports.

In order to consider that an ITI can be used as a decoupling and power dividing circuit, we need the network shown in Fig. 2 which is constituted with an (m,n)-port ITI and n conductances.

1) The case where all output ports are terminated in equal conductances G_o, then

$$I_o = G_o V_o. \tag{6}$$

Let's make use of the ITI satisfying

$$[T] = [P \cdots P_m] \tag{7}$$

where $P_1, \cdots, P_m$ are orthogonal vectors, and

$$P_k = [p_{1k} \cdots p_{nk}]^t, \qquad (k = 1, \cdots, m).$$

Substituting (6) and (7) into (5) yields

$$V_a = G_o^{-1} \begin{bmatrix} P_1^t \\ \cdot \\ \cdot \\ \cdot \\ P_m^t \end{bmatrix} [P_1 \cdots P_m] I_a. \tag{8}$$

That is, the input ports are decoupled from one another. In this case, the input current I_{ka} from the input port k is transformed as

$$P_k I_{ka} = I_o. \tag{9}$$

By using (6) and (9), one can get

$$I_{1o}V_{1o} : \cdots : I_{no}V_{no} = p_{1k}^2 : \cdots : p_{nk}^2. \tag{10}$$

That is, the input power from an input port k is distributed among the n output ports in the proportion of (10). The vectors $P_1, \cdots, P_m$ may not necessarily be orthonormal. However, for the sake of simplicity, let them be orthonormal vectors

$$P_i^t P_j = \begin{cases} 0, & (i \neq j) \\ 1, & (i = j) \end{cases} \quad (i,j = 1, \cdots, m). \tag{11}$$

2) Next consider the case where all output ports are not always terminated in equal conductances; that is, the case where the ports are terminated in conductances $a_1^2 G_o, \cdots, a_n^2 G_o$, then

$$I_o = [A]^2 G_o V_o \tag{12}$$

where $[A] = \text{diag}\,[a_1, \cdots, a_n]$.

Let's make use of the ITI satisfying

$$[T] = [A][P_1' \cdots P_m'] \tag{13}$$

where $P_1', \cdots, P_m'$ are orthonormal vectors, and

$$P_k' = [p_{1k}' \cdots p_{nk}']^t, \qquad (k = 1, \cdots, m)$$

then the input ports are decoupled from one another and the input power from an input port k is distributed among the n output ports in the proportion

$$p_{1k}'^2 : \cdots : p_{nk}'^2. \tag{14}$$

In this case, $a_1, \cdots, a_n$ can be chosen arbitrarily. However, for the sake of simplicity, let them satisfy the following relations:

$$[A] = \text{diag}\,[a_1, \cdots, a_n] \tag{15}$$

where

$$a_k > 0, \quad (k = 1, \cdots, n) \quad \text{and} \quad a_1^2 + \cdots + a_n^2 = 1.$$

C. TEM Modes on a Multiwire Line

An n-wire line inside a shielding conductor or over a ground plane supports n independent TEM modes. This paper describes how one can determine independent TEM modes on a n-wire line. We suppose that a characteristic admittance matrix $[\eta]$ of the n-wire line is given.

A TEM mode is determined with a voltage vector and a current vector on the n-wire line. So we suppose the n-wire line supports a TEM mode (ith TEM model) with voltages $U_{1i}, \cdots, U_{ni}$ and currents $J_{1i}, \cdots, J_{ni}$ on the wires as shown in Fig. 3, and ratios $U_{\mu i}/U_{\nu i}$ and $J_{\mu i}/J_{\nu i} (\mu, \nu = 1, \cdots, n)$ are given by real numbers, respectively.

The orthogonality condition among TEM modes can be given in the following definition [8].

Definition 1: The condition that any two of such TEM modes (ith and jth) are orthogonal or independent, is given by

$$U_i^t J_j \begin{cases} = 0, & (i \neq j) \\ \neq 0, & (i = j) \end{cases} \tag{16}$$

where

$$U_i^t = [U_{1i} \cdots U_{ni}]$$
$$J_i^t = [J_{1i} \cdots J_{ni}].$$

(End of definition.)

If the n-wire line is terminated on its output side (right-hand side of Fig. 1) with a conductive circuit $[G]$ equal to $[\eta]$, then the voltage vector U_o and the current vector J_o on the input side satisfy

$$J_o = [\eta] U_o = [G] U_o. \tag{17}$$

The matrix $[\eta]$ has n eigenvalues $\alpha_1, \cdots, \alpha_n$ which are

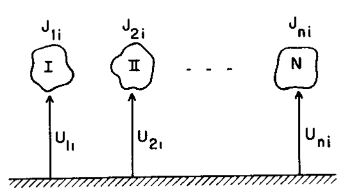

Fig. 3. Currents and voltages for ith TEM mode on n-wire line.

all real and positive, because the matrix is hyperdominant. The eigenvectors corresponding to the eigenvalues can be obtained instantly. If α_i is the l-ple eigenvalue, then there are l linear independent eigenvectors corresponding to α_i. These l vectors can be made to constitute orthonormal eigenvectors by the Schmidt process [10]. Thus one can obtain a set of n orthonormal eigenvectors $\boldsymbol{P}_1,\cdots,\boldsymbol{P}_n$. The eigenvalues and the eigenvectors satisfy

$$[\eta]\boldsymbol{P}_i = \alpha_i \boldsymbol{P}_i, \qquad (i = 1,\cdots,n). \tag{18}$$

Consider a network which is constituted with an ITI, the n-wire line, and the conductive circuit $[G]$ shown in Fig. 4. Let's suppose that the ITI is composed with the eigenvectors $\boldsymbol{P}_1,\cdots,\boldsymbol{P}_n$. That is,

$$[T] = [\boldsymbol{P}_1 \cdots \boldsymbol{P}_n]. \tag{19}$$

The kth input voltage V_{ka} and current I_{ka} are transformed, respectively, into a voltage vector $\boldsymbol{U}_k$ and a current vector $\boldsymbol{J}_k$ on the n-wire line by the vector $\boldsymbol{P}_k$ as

$$\boldsymbol{U}_k = \boldsymbol{P}_k V_{ka} \qquad \boldsymbol{J}_k = \boldsymbol{P}_k I_{ka}, \qquad (k = 1,\cdots,n). \tag{20}$$

Let's consider two TEM modes, with voltage vectors $\boldsymbol{U}_i$ and $\boldsymbol{U}_j$, and current vectors $\boldsymbol{J}_i$ and $\boldsymbol{J}_j$, which are excited with ith and jth input ports by using transformation vectors $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$, respectively. As $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ are orthonormal vectors, $\boldsymbol{U}_i$ and $\boldsymbol{J}_j$ satisfy

$$\boldsymbol{U}_i{}^t \boldsymbol{J}_j = \boldsymbol{P}_i{}^t \boldsymbol{P}_j V_{ia} I_{ja} \begin{cases} = 0, & (i \neq j) \\ \neq 0, & (i = j) \end{cases} \tag{21}$$

so that $\boldsymbol{U}_i$ and $\boldsymbol{J}_j$ satisfy the condition of orthogonality of the TEM modes, as given in Definition 1.

Definition 2: The set of TEM modes that satisfy (20) is defined to be a set of basic TEM modes.

(End of definition.)

Consider the case where each wire of the n-wire line supports only a kth basic TEM mode. By using (17), the voltage vector $\boldsymbol{U}_k$ and the current vector $\boldsymbol{J}_k$ of the kth TEM mode satisfy

$$\boldsymbol{J}_k = [\eta]\boldsymbol{U}_k. \tag{22}$$

By using (18), (20), and (22), one has

$$\boldsymbol{J}_k = [\eta]\boldsymbol{U}_k = [\eta]\boldsymbol{P}_k V_{ka} = \alpha_k \boldsymbol{P}_k V_{ka}, \qquad \therefore \boldsymbol{J}_k = \alpha_k \boldsymbol{U}_k. \tag{23}$$

This equation can be rewritten as

$$\begin{aligned} J_{1k} &= \alpha_k U_{1k} \\ &\vdots \\ J_{nk} &= \alpha_k U_{nk}. \end{aligned} \tag{24}$$

Therefore, if a TEM mode on the n-wire line is determined, then we can define "characteristic wire admittances" for each wire of the n-wire line. In the case of the basic TEM mode, the characteristic wire admittances are all equal to α_k.

Equation (23) is very similar to (6). The differences

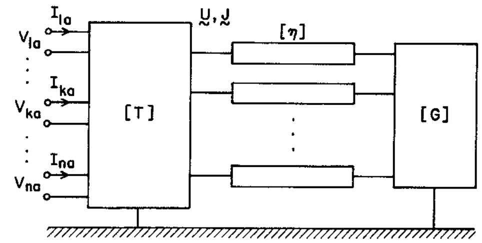

Fig. 4. Network constituted with ITI, n-wire line, and conductive circuit.

between them are the conductances G_o of (6) and the characteristic wire admittances α_k. So the n-wire line network supporting the basic TEM modes is almost identical to the network which is explained in 1) of Section II-B.

D. Modified TEM Modes

We describe the one that can introduce TEM modes other than the basic TEM modes in this section. Take the positive diagonal matrix $[A]$ defined by (15), and let's take up a new matrix $[A]^{-1}[\eta][A]^{-1}$. This matrix has n eigenvalues $\beta_1,\cdots,\beta_n$ which are all real and positive, because the matrix $[\eta]$ is hyperdominant. We can obtain n orthonormal eigenvectors $\boldsymbol{P}_1',\cdots,\boldsymbol{P}_n'$ of the matrix corresponding to the eigenvalues. Then the following relationships are satisfied:

$$[A]^{-1}[\eta][A]^{-1}\boldsymbol{P}_k' = \beta_k \boldsymbol{P}_k', \qquad (k = 1,\cdots,n). \tag{25}$$

Consider the network shown in Fig. 4. In this case, let the transformation matrix $[T]$ of the ITI be presented as

$$[T] = [A][\boldsymbol{P}_1' \cdots \boldsymbol{P}_n']. \tag{26}$$

The kth input voltage V_{ka} and current I_{ka} are transformed, respectively, into a voltage vector $\boldsymbol{U}_k'$ and a current vector $\boldsymbol{J}_k'$ on the n-wire line by the transformation matrix as

$$\boldsymbol{J}_k' = [A]\boldsymbol{P}_k' I_{ka} \tag{27a}$$

$$\boldsymbol{U}_k' = [A]^{-1}\boldsymbol{P}_k' V_{ka}, \qquad (k = 1,\cdots,n) \tag{27b}$$

so the ith voltage vector $\boldsymbol{U}_i'$ and the jth current vector $\boldsymbol{J}_j'$ satisfy

$$\boldsymbol{U}_i'{}^t \boldsymbol{J}_j' \begin{cases} = 0, & (i \neq j) \\ \neq 0, & (i = j). \end{cases} \tag{28}$$

That is, the set of voltage vectors $\boldsymbol{U}_k'$ and current vectors $\boldsymbol{J}_k'(k = 1,\cdots,n)$ satisfy the orthogonal condition of the TEM mode, given by Definition 1.

Definition 3: A set of TEM modes that satisfy (27) is called a set of modified TEM modes.

(End of definition.)

By using (17), (25), and (27), $\boldsymbol{J}_k'$ and $\boldsymbol{U}_k'$ should satisfy

$$\boldsymbol{J}_k' = [\eta]\boldsymbol{U}_k' = [\eta][A]^{-1}\boldsymbol{P}_k' V_{ka} = [A]\beta_k \boldsymbol{P}_k' V_{ka}, \qquad \therefore \boldsymbol{J}_k' = \beta_k [A]^2 \boldsymbol{U}_k'. \tag{29}$$

This equation can be rewritten as

$$J_{1k}' = \beta_k a_1^2 U_{1k}'$$
$$\vdots$$
$$J_{nk}' = \beta_k a_n^2 U_{nk}'. \qquad (30)$$

That is, if a modified TEM mode on the n-wire line is determined by using the matrices $[\eta]$ and $[A]$, the characteristic wire admittances for each wire are not necessarily equal corresponding to $a_1, \cdots, a_n$.

Equation (29) is very similar to (12). The differences between them are the value of conductance G_o of (12) and the eigenvalue β_k. So the n-wire line network supporting the modified TEM modes is almost identical to the network which is explained in 2) of Section II-B.

The matrix $[A]$ can be chosen arbitrarily under the condition given by (15). Therefore, it is shown that the n TEM modes on the n-wire line are determined by the geometrical arrangements of the n wires (which means the characteristic admittance matrix $[\eta]$) as well as by the manner of excitation into the n-wire line (which means the choice of $[A]$).

III. POWER DIVIDING RATIO AND DECOUPLING

In the previous section, we considered the TEM modes on an n-wire line, and it was found that the n-wire line excited with a TEM mode may be utilized in making a power divider. In this section, let us take a two-wire line, for example, and describe how to choose the input and output admittances for the two-wire circuit shown in Fig. 5.

The characteristic admittance matrix $[\eta]$ and the characteristic impedance matrix $[\zeta]$ of a two-wire line can be expressed as

$$[\eta] = \begin{bmatrix} \eta_{11} & -\eta_{12} \\ -\eta_{12} & \eta_{22} \end{bmatrix} \qquad (31a)$$

$$[\zeta] = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} = \frac{1}{\eta_{11}\eta_{22} - \eta_{12}^2} \begin{bmatrix} \eta_{22} & \eta_{12} \\ \eta_{12} & \eta_{11} \end{bmatrix}. \qquad (31b)$$

Now we try to divide the power from an input port between two output ports on different terminating admittances by utilizing a modified TEM mode on the two-wire line. Let's suppose that the ratios among the terminating admittances are in the proportion of real numbers. So the admittances are assumed to be $a_1^2 Y_l(\lambda)$ and $a_2^2 Y_l(\lambda)$, respectively, as shown in Fig. 5. Then

$$I_{1l} = a_1^2 Y_l(\lambda) V_{1l} \qquad I_{2l} = a_2^2 Y_l(\lambda) V_{2l}. \qquad (32)$$

In the case of the modified TEM modes, the eigenvalues and the orthonormal eigenvectors of $[A]^{-1}[\eta][A]^{-1}$ come into question as described in Section II-D. The matrix $[A]$ should be diag $[a_1, a_2]$. Let the eigenvalues be denoted by β_1 and β_2, then these satisfy

$$\beta_1 + \beta_2 = a_1^{-2}\eta_{11} + a_2^{-2}\eta_{22} \qquad (33a)$$
$$\beta_1\beta_2 = a_1^{-2}a_2^{-2}(\eta_{11}\eta_{22} - \eta_{12}^2). \qquad (33b)$$

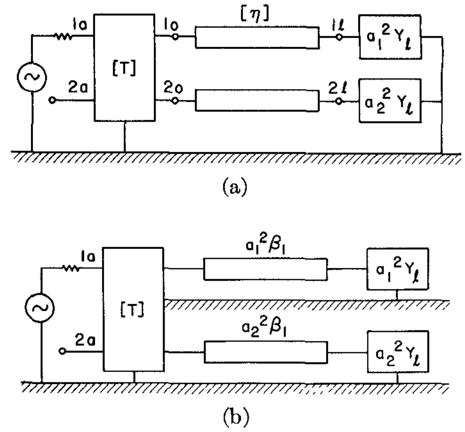

Fig. 5. (a) Network constituted with ITI, two-wire line, and admittances for output ports. (b) Its equivalent network.

Let the orthonormal eigenvectors be denoted by $\boldsymbol{P}_1' = [p_1' p_2']^t$ and $\boldsymbol{P}_2' = [p_2' \; - p_1']^t$ corresponding to β_1 and β_2, respectively, and by using (25), one obtains the following equations:

$$\begin{bmatrix} \eta_{11} & -\eta_{12} \\ -\eta_{12} & \eta_{22} \end{bmatrix} \begin{bmatrix} p_1'/a_1 \\ p_2'/a_2 \end{bmatrix} = \beta_1 \begin{bmatrix} a_1 p_1' \\ a_2 p_2' \end{bmatrix} \qquad (34a)$$

$$\begin{bmatrix} \eta_{11} & -\eta_{12} \\ -\eta_{12} & \eta_{22} \end{bmatrix} \begin{bmatrix} p_2'/a_1 \\ -p_1'/a_2 \end{bmatrix} = \beta_2 \begin{bmatrix} a_1 p_2' \\ a_2 p_1' \end{bmatrix} \qquad (34b)$$

where $p_1'^2 + p_2'^2 = 1$.

The transformation matrix $[T]$ of the ITI should be chosen as

$$[T] = [A][\boldsymbol{P}_1' \boldsymbol{P}_2']. \qquad (35)$$

By using (5) and (35), one can get

$$\begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} = \begin{bmatrix} a_1 p_1' & a_2 p_2' \\ a_1 p_2' & -a_2 p_1' \end{bmatrix} \begin{bmatrix} V_{1o} \\ V_{2o} \end{bmatrix} \qquad (36a)$$

$$\begin{bmatrix} a_1 p_1' & a_1 p_2' \\ a_2 p_2' & -a_2 p_1' \end{bmatrix} \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} = \begin{bmatrix} I_{1o} \\ I_{2o} \end{bmatrix}. \qquad (36b)$$

The transmission equations of the two-wire line can be written

$$\begin{bmatrix} V_{1o} \\ V_{2o} \end{bmatrix} = c \begin{bmatrix} V_{1l} \\ V_{2l} \end{bmatrix} + s \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12} & \zeta_{22} \end{bmatrix} \begin{bmatrix} I_{1l} \\ I_{2l} \end{bmatrix} \qquad (37a)$$

$$\begin{bmatrix} I_{1o} \\ I_{2o} \end{bmatrix} = s \begin{bmatrix} \eta_{11} & -\eta_{12} \\ -\eta_{12} & \eta_{22} \end{bmatrix} \begin{bmatrix} V_{1l} \\ V_{2l} \end{bmatrix} + c \begin{bmatrix} I_{1l} \\ I_{2l} \end{bmatrix}. \qquad (37b)$$

Now an input is assumed to be incident on port $1a$. The input current from port $1a$ becomes transformed by the ITI as

$$a_1 p_1' I_{1a} = I_{1o} \qquad a_2 p_2' I_{1a} = I_{2o}. \qquad (38)$$

Substituting (32), (34a), and (38) into (37b) yields

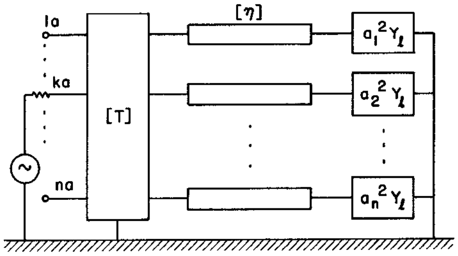

Fig. 6. Network constituted with ITI, n-wire line, and admittances for output ports.

$$\frac{V_{1l}}{I_{1a}} = \frac{p_1'}{a_1(s\beta_1 + cY_l(\lambda))} \qquad \frac{V_{2l}}{I_{1a}} = \frac{p_2'}{a_2(s\beta_1 + cY_l(\lambda))}. \tag{39}$$

Substituting (39) into (37a), and by using (36a),

$$\frac{V_{1a}}{I_{1a}} = \frac{c\beta_1 + sY_l(\lambda)}{\beta_1(s\beta_1 + cY_l(\lambda))} \qquad \frac{V_{2a}}{I_{1a}} = 0. \tag{40}$$

Thus the two input ports (port $1a$ and $2a$) of the circuit shown in Fig. 5 are decoupled, and the input impedance looked in from the port $1a$ can be obtained by (40).

By using (32) and (39), the ratio of power division of the output ports can be obtained as

$$\frac{V_{1l}I_{1l}}{V_{2l}I_{2l}} = \frac{p_1'^2}{p_2'^2} \tag{41}$$

and is constant for all frequencies. Therefore, the circuit consisting of an ITI and a two-wire line becomes a power divider having a constant power division ratio at all frequencies.

In (40), if the value of an admittance $Y_l(\lambda)$ is equal to a value of conductance, and moreover

$$Y_l = \beta_1 \tag{42}$$

then the input impedance looking in from the port $1a$ becomes resistance $1/\beta_1$, so the circuit can be matched at port $1a$ for all frequencies.

These results can be easily extended to a circuit having n output ports. That is, consider the characteristic admittance matrix given by (2) and the diagonal matrix $[A]$ given by (15). Next one obtains β_k and $P_k'(k = 1,\cdots,n)$ by using the method described in Section II-D. If one terminates admittances $a_1^2Y_l(\lambda),\cdots,a_n^2Y_l(\lambda)$, respectively, to the output ports $1l,\cdots,nl$ shown in Fig. 6, then the input ports $1a,\cdots,na$ are decoupled from each other and the ratio of power division to the output ports is obtained as given by (14).

IV. EVEN AND ODD TEM MODES ON THE n-WIRE LINE

It is well known that even and odd TEM modes exist on a coupled line or two-wire line. If the two-wire line supports only an odd TEM mode, current doesn't flow on a shielding conductor (or over a ground plane). This section describes how one can excite $n - 1$ TEM modes, each of which resembles the odd mode, excluding only a TEM mode on an n-wire line.

A. Even and Odd TEM Modes for Basic TEM Modes

The characteristic admittance matrix $[\eta]$ of an n-wire line can be represented by (2). Elements η_{ie} $(i = 1,\cdots,n)$ of this matrix are defined by (3).

The reduction polynomial for obtaining eigenvalues of $[\eta]$ can be presented as follows:

$$|[\eta] - \alpha E_n| = \begin{vmatrix} \eta_{1e} - \alpha & -\eta_{12} & \cdots & -\eta_{1n} \\ \eta_{2e} - \alpha & \eta_{22} - \alpha & \cdots & -\eta_{2n} \\ \vdots & \vdots & \ddots & \\ \eta_{ne} - \alpha & -\eta_{2n} & \cdots & \eta_{nn} - \alpha \end{vmatrix} \tag{43}$$

where E_n: $n \times n$ unit matrix.

If every η_{ie} $(i = 1,\cdots,n)$ is equal to each other, then the reduction polynomial has a root (an eigenvalue) α_u which satisfies

$$\alpha_u = \eta_{1e} = \cdots = \eta_{ne}. \tag{44}$$

The orthonormal eigenvector P_u corresponding to this eigenvalue should be given as

$$P_u^t = \frac{1}{n^{1/2}}[1\cdots1]. \tag{45}$$

There are $n - 1$ orthonormal eigenvectors which are different from P_u. Let such eigenvectors be represented as

$$P_k^t = [p_{1k}\cdots p_{nk}], \qquad (k = 1,\cdots,n-1). \tag{46}$$

Since P_u and P_k $(k = 1,\cdots,n-1)$ are all orthonormal eigenvectors,

$$\sum_{j=1}^{n} p_{jk} = 0, \qquad (k = 1,\cdots,n-1) \tag{47}$$

should be satisfied for P_k.

The current vector J_k on the n-wire line excited by the transformation vector P_k is represented by (20). So the currents $J_{1k},\cdots,J_{nk}$ on the n wires should satisfy

$$J_{1k} + J_{2k} + \cdots + J_{nk} = 0, \qquad (k = 1,\cdots,n-1). \tag{48}$$

That is, the resultant return current does not flow on a shielding conductor for the basic TEM mode corresponding to P_k $(k = 1,\cdots,n-1)$. All of these $n - 1$ basic TEM modes are similar to an odd mode on a two-wire line. On the other hand the basic TEM mode excited with P_u is similar to an even mode, so we call the TEM mode an even mode.

B. Even and Odd TEM Modes for Modified TEM Modes

We consider the case where $\eta_{1e},\cdots,\eta_{ne}$ are not always equal to each other. So let's assume that $\eta_{1e},\cdots,\eta_{ne}$ satisfy

$$\eta_{ke} = a_k^2\eta_e, \qquad (k = 1,\cdots,n) \tag{49}$$

where η_e is positive constant.

Let's take a diagonal matrix $[A]$ given by (15), and take up a new matrix $[A]^{-2}[\eta]$. The reduction polynomial for obtaining eigenvalues of $[A]^{-2}[\eta]$ should be presented as follows:

$$| [A]^{-2}[\eta] - \beta E_n | = \begin{bmatrix} a_1^{-2}\eta_{1e} - \beta & -a_1^{-2}\eta_{12} & \cdots & -a_1^{-2}\eta_{1n} \\ a_2^{-2}\eta_{2e} - \beta & a_2^{-2}\eta_{22} - \beta & \cdots & -a_2^{-2}\eta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_n^{-2}\eta_{ne} - \beta & -a_n^{-2}\eta_{2n} & \cdots & a_n^{-2}\eta_{nn} - \beta \end{bmatrix}. \tag{50}$$

By substituting (49) into (50), one can obtain an eigenvalue β_u which satisfies

$$\beta_u = \eta_e. \tag{51}$$

Let other $n - 1$ eigenvalues be denoted by β_k, $(k = 1,\cdots,n - 1)$.

The orthonormal eigenvector $\boldsymbol{Q}_u$ corresponding to the eigenvalue β_u should be given as

$$\boldsymbol{Q}_u{}^t = \frac{1}{n^{1/2}}[1\cdots1]. \tag{52}$$

Let other $n - 1$ orthonormal eigenvectors be denoted by $\boldsymbol{Q}_k$ $(k = 1,\cdots,n - 1)$.

The matrix $[A]^{-2}[\eta]$ can be represented as

$$[A]^{-2}[\eta] = [A]^{-1}([A]^{-1}[\eta][A]^{-1})[A]. \tag{53}$$

Therefore, the reduction polynomials for $[A]^{-2}[\eta]$ and $[A]^{-1}[\eta][A]^{-1}$ are invariant to each other. That is, the eigenvalues of $[A]^{-1}[\eta][A]^{-1}$ also are β_u and β_k $(k = 1,\cdots,n - 1)$. So the orthonormal eigenvectors $\boldsymbol{P}_u'$ and $\boldsymbol{P}_k'$ of $[A]^{-1}[\eta][A]^{-1}$ must satisfy the following equations:

$$\boldsymbol{Q}_u = [A]^{-1}\boldsymbol{P}_u'$$
$$\boldsymbol{Q}_k = [A]^{-1}\boldsymbol{P}_k', \qquad (k = 1,\cdots,n - 1). \tag{54}$$

The voltage vectors $\boldsymbol{U}_k'$ on the n-wire line excited with modified TEM modes are represented by (27b). Let a voltage vector $\boldsymbol{U}_u'$ on the n-wire line be excited by using the transformation vector $[A]^{-1}\boldsymbol{P}_u'$, then the following relationships among n voltages on the n wires satisfy

$$U_{1u}' = U_{2u}' = \cdots = U_{nu}' \tag{55}$$

where

$$\boldsymbol{U}_u'^t = [U_{1u}'\cdots U_{nu}'].$$

The TEM mode whose voltages on the n-wire line are given by (55), is similar to an even mode, so we call the TEM mode an even mode.

The orthogonality condition among TEM modes can be given by Definition 1 or (16). Therefore, $n - 1$ current vectors excluding the even mode must satisfy the following equations:

$$J_{1k}' + J_{2k}' + \cdots + J_{nk}' = 0, \qquad (k = 1,\cdots,n - 1) \tag{56}$$

where

$$J_k'^t = [J_{1k}'\cdots J_{nk}'].$$

That is, all of these $n - 1$ modified TEM modes are similar to an odd mode.

V. CONCLUSION

This paper has presented a restatement of mathematical consideration of TEM modes on the n-wire line deduced from the characteristic admittance matrix. The following results have been shown conclusively: 1) n independent TEM modes can be determined by obtaining eigenvectors on the n-wire line; 2) the TEM modes are determined by the geometrical arrangement of the n wires as well as by the manner of excitation on the n-wire line; 3) power division ratios on each wire; 4) ratios among terminating admittances for output ports of each wire; and 5) one can excite one TEM mode similar to an even mode and $n - 1$ TEM modes, each of which resembles an odd mode, on the n-wire line. These results may be applied to establish synthesizing methods for broad-band n-way hybrid power dividers with arbitrary output signals.

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